UDC 531.36

# THE STABILITY OF SOLUTIONS OF CERTAIN SYSTEMS OF DIFFERENTIAL EQUATIONS WITH A SMALL PARAMETER AT DERIVATIVES

PMM Vol. 41, № 3, 1977, pp. 567-573
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(Received April 26, 1977)

The problem of nonasymptotic stability of the zero solution of certain systems of differential equations with a small parameter at derivatives is considered. Conditions are derived for which the stability of zero solution of the degenerate (with respect to the small parameter) system entails also the stability of zero solution of the input system. The problem of stability of stabilization by gyroscopic systems with considerable intrinsic angular momenta is used as an example.

1. Let the motion of some system be defined by the differential equations

$$dz_k / dt = Z_k(t, \mu, z_j, x_i) \quad (k = 1, \ldots, m) \tag{1.1}$$

$$\frac{dx_k}{dt} = \sum_{j=1}^{n} p_{kj}(\mu) x_j + X_k(t, \mu, z_j, x_i) \quad (k = 1, \ldots, \xi)$$

$$\mu \frac{dx_k}{dt} = \sum_{j=1}^{n} p_{kj}(\mu) x_j + X_k(t, \mu, z_j, x_i) \quad (k = \xi + 1, \ldots, r)$$

$$\mu^2 \frac{dx_k}{dt} = \sum_{j=1}^{n} p_{kj}(\mu) x_j + X_k(t, \mu, z_j, x_i) \quad (k = r + 1, \ldots, n)$$

where $\mu$ is a small parameter; $p_{kj}(\mu)$ are continuous functions of $\mu$; and $Z_k$ and $X_k$ are functions holomorphic over the set of variables $z_j$ and $x_i$ $(j = 1, \ldots, m;\ i = 1, \ldots, n)$ whose expansions do not contain terms of power lower than the second, whose coefficients are continuous bounded functions of $t$ and $\mu$ and

$$Z_k(t, \mu, z_j, 0) = 0 \quad (k = 1, \ldots, m), \quad X_k(t, \mu, z_j, 0) = 0 \quad (k = 1, \ldots, n)$$

We assume that the above properties apply in a certain region.

The zero solution of system (1.1) is stable (nonasymptotically) by Liapunov's theorem, provided that all roots of the equation

$$\Delta(\lambda) = \begin{vmatrix} \|\delta_{kj}\lambda - p_{kj}(\mu)\| \\ \cdots\cdots\cdots \\ \|\mu\delta_{kj}\lambda - p_{kj}(\mu)\| \\ \cdots\cdots\cdots \\ \|\mu^2\delta_{kj}\lambda - p_{kj}(\mu)\| \end{vmatrix} \begin{matrix} 1 \\ \vdots \\ \xi \\ \vdots \\ r \\ \vdots \\ n \end{matrix} = 0 \qquad (1.2)$$

have negative real parts, i.e., if Eq.(1.2) satisfies Hurwitz conditions. The degenerate system for Eqs.(1.1) is of the form

$$\begin{aligned} & dz_k / dt = Z_k(t, 0, z_j, x_i) \quad (k = 1, \ldots, m) \\ & dx_k/dt = \sum_{j=1}^{n} p_{kj}x_j + X_k(t, 0, z_j, x_i) \quad (k = 1, \ldots, \xi) \\ & 0 = \sum_{j=1}^{n} p_{kj}x_j + X_k(t, 0, z_j, x_i) \quad (k = \xi + 1, \ldots, n) \\ & p_{kj} = p_{kj} \quad (\mu = 0) \end{aligned} \qquad (1.3)$$

The characteristic equation of the first approximation system (1.3) has $m$ zero roots, and the remaining $\xi$ of its roots are determined by the equation

$$\begin{vmatrix} \|\delta_{kj}\lambda - p_{kj}\| \\ \cdots\cdots \\ \|p_{kj}\| \end{vmatrix} \begin{matrix} 1 \\ \vdots \\ \xi \\ \vdots \\ n \end{matrix} = 0 \qquad (1.4)$$

Let us investigate the conditions under which the stability of the zero solution of the degenerate system(1.3) implies the stability of zero solution of the input system (1.1). The case of asymptotic stability in a similar problem was considered in [2-4]. The following theorem is valid.

Theorem. If all roots of the characteristic equation of the degenerate system (1.3), except the $m$ zero roots, have negative real parts and the equations

$$\begin{vmatrix} \|\delta_{kj}\alpha - p_{kj}\| \\ \cdots\cdots \\ \|p_{kj}\| \end{vmatrix} \begin{matrix} \xi+1 \\ \vdots \\ r \\ \vdots \\ n \end{matrix} = 0 \qquad (1.5)$$

$$\begin{vmatrix} \|\delta_{kj}\beta - p_{kj}\| \end{vmatrix} \begin{matrix} r+1 \\ \vdots \\ n \end{matrix} = 0 \qquad (1.6)$$

satisfy Hurwitz conditions, and among the roots of Eq. (1.5) there are no multiple roots, the zero solution of the degenerate system (1.3) is stable, and for fairly small parameters. $\mu$ the zero solution of system (1.1) is, also, stable.

System (1.3) admits $m$ independent holomorphic Liapunov integrals

$$z_k + \varphi_k(z_1, \ldots, z_m, x_1, \ldots, x_\xi, t) = A_k \quad (k = 1, \ldots, m) \qquad (1.7)$$

where $\varphi_k$ is a function that is holomorphic over the set of variables $z_j, x_1, \ldots, x_\xi$ whose expansion does not contain terms of order lower than the second, which vanishes when $x_1 = 0, x_2 = 0, \ldots, x_\xi = 0$ and whose coefficients are bounded functions of $t$.

Proof. Let us consider the degenerate system (1.3). By the theorem on implicit functions the system of $(n - \xi)$ algebraic equations

$$0 = \sum_{j=1}^{n} p_{kj} x_j + X_k(t, 0, z_j, x_i) = f_k(t, z_j, x_i) \quad (k = \xi + 1, \ldots, n) \tag{1.8}$$

admits the unique solution

$$x_s = x_s(t, z_j, x_1, \ldots, x_\xi) \quad (s = \xi + 1, \ldots, n) \tag{1.9}$$

in the form of holomorphic functions of variables $x_1, x_2, \ldots, x_\xi$ which vanish when $x_1 = 0, x_2 = 0, \ldots, x_\xi = 0$ whose coefficients are continuous bounded functions of time and critical variables $z_j$ if the Jacobian $|df_k / dx_j|_{k, j=\xi+1, \ldots, n}$ is nonzero when $x_i = 0$ $(i = 1, \ldots, n)$. For system (1.8) that Jacobian is equal $|p_{kj}|_{\xi+1}^{n}$ and under conditions of the theorem it is nonzero.
Substituting solution (1.9) into the first $(m + \xi)$ equations of system (1.3) we obtain for the latter

$$\begin{aligned} dz_k / dt &= Z_k'(t, z_j, x_1, \ldots, x_\xi) \quad (k = 1, \ldots, m) \\ \frac{dx_k}{dt} &= \sum_{j=1}^{\xi} p_{kj}' x_j + X_k'(t, z_j, x_1, \ldots, x_\xi) \quad (k = 1, \ldots, \xi) \end{aligned} \tag{1.10}$$

Owing to the method of forming functions $Z_k'$ and $X_k'$ vanish when $x_1 = 0, \ldots$ $x_\xi = 0$. System (1.10) belongs to the kind of Liapunov's system [1]. Using Liapunov's theorem we obtain that under conditions considered here its zero solution is stable, and that integrals of the form (1.7) exist.

Let us consider Eq. (1.2), and show that under conditions of the theorem the roots of that equation separate for $\mu \to 0$ into $\xi$ roots which tend to the values of roots of Eq.(1.4) and at the limit are equal to them, while the remaining $(n - \xi)$ roots tend to $\infty$. The $(r - \xi)$ roots of the second group can be represented in the form $\lambda(\mu) = \alpha(\mu) / \mu$, where $\alpha(\mu) \to \alpha_0$ when $\mu \to 0$ ($\alpha_0$ is the root of Eq.(1.5)); the remaining $(n - r)$ roots are of the form $\lambda(\mu) = \beta(\mu) / \mu^2$ where $\beta(\mu) \to \beta_0$ when $\mu \to 0$ ($\beta_0$ is the root of Eq.(1.6)).

We multiply Eq.(1.2) by $\rho^n$ and introduce in it the new variables $\rho = 1 / \lambda$. In the obtained equation $d(\rho, \mu) = 0$ the roots $\rho = \rho(\mu)$ are continuous functions of parameter $\mu$ when $|p_{kj}(\mu)|^n_1 \neq 0$. By the theorem on the roots of an algebraic equation we find that when $\mu \to 0$ $\xi$ roots $\rho$ tend to the values of roots of the degenerate equation $d(\rho, 0) = 0$ and the remaining tend to zero. Reverting to the variable $\lambda$. we obtain the first part of the statement formulated above.

We multiply Eq. (1.2) by $\mu^\xi$ and set it $\mu\lambda = \alpha$. We write the obtained equation $d_1(\alpha, \mu) = 0$ in the form

$$d_1(\alpha, \mu) = F(\alpha) + \mu F_1(\alpha, \mu) = 0 \qquad (1.11)$$

$$F(\alpha) = \alpha^{\xi} \left| \begin{array}{c} \|\delta_{kj}\alpha - p_{kj}\| \\ \cdots\cdots\cdots \\ \|-p_{kj}\| \end{array} \right| \begin{array}{c} \xi+1 \\ \vdots \\ r \\ \vdots \\ n \end{array}$$

We introduce the notation $\Delta\alpha = \alpha - \alpha_0$, where $\alpha$ is the root of Eq.(1.11) and $\alpha_0$ is the nonzero solution of equation $F(\alpha) = 0$, i.e. the root of Eq. (1.5).

By expanding (1.11) in series in the neighborhood of $\alpha_0$, we obtain an equation which for $dF(\alpha_0)/d\alpha \neq 0$ and small $\mu$, determines by the theorem on implicit functions the unique holomorphic function $\Delta\alpha = \Delta\alpha(\mu)$ and vanishes when $\mu = 0$. Hence for fairly small $\mu$ the related roots $\alpha(\mu)$ are close to $\alpha_0$ and, when $\mu \to 0$ $\alpha(\mu) \to \alpha_0$.

We multiply Eq.(1.2) by $\mu^{r+\xi}$ and set $\mu^2\lambda = \beta$. The theorem on the continuous dependence of roots of algebraic equations on their coefficients implies that $(n - r)$ roots of the obtained equations $d_2(\beta, \mu) = 0$ $\beta(\mu) \to \beta_0$ when $\mu \to 0$. The second part of the statement is proved.

Thus, when Eqs. (1.4)-(1.6) satisfy the Hurwitz conditions, the real parts of all roots of Eq. (1.2) are negative when parameter $\mu$ is fairly small. The zero solution of system (1.1) is then Liapunov stable. The theorem is proved.

**2.** As an example of the theorem application, we shall consider the problem of stability of the established motion of a system of gyroscopic stabilization. We have Liapunov's critical case.

For the model of a system of gyroscopic stabilization considered in [5, 6] the differential equations of perturbed motion are (in the notation of those papers) of the form

$$(2.1)$$

$$\frac{d}{dt}\sum_{j=1}^{n} a_{kj}q_j^{\boldsymbol{\cdot}} + \sum_{j=1}^{n}(b_{kj}^{\circ} + g_{kj}^{\circ})\, q_j^{\boldsymbol{\cdot}} = Q_k' + Q_k'' \qquad (k = 1, \ldots, n)$$

$$\frac{d}{dt}\sum_{j=n+1}^{n+u} L_{kj}q_j^{\boldsymbol{\cdot}} + \sum_{j=m+1}^{s} B_{kj}^{\circ}q_j^{\boldsymbol{\cdot}} + \sum_{j=n+1}^{n+u} R_{kj}^{\circ}q_j^{\boldsymbol{\cdot}} = Q_k' + Q_k''$$

$$(k = n+1, \ldots, n+u)$$

$$dq_k / dt = q_k^{\boldsymbol{\cdot}} \qquad (k = 1, \ldots, n)$$

The first $n$ equations correspond to generalized mechanical coordinates, the following $u$ equations correspond to generalized electrical coordinates; $q_j^{\boldsymbol{\cdot}}$ $(j = n+1, \ldots, n+u)$ are mesh currents

$$Q_k' = 0 \qquad (k = 1, \ldots, m;\ n + \mu + 1, \ldots, n + u),$$

$$Q_k' = \sum_{j=n+1}^{n+u} A_{kj}^{\circ}q_j^{\boldsymbol{\cdot}} \qquad (k = m+1, \ldots, s)$$

$$Q_k' = -\sum_{j=s+1}^{n} c_{kj}q_j \quad (k = s+1, \ldots, n),$$

$$Q_k' = -\sum_{j=1}^{l} \omega_{kj}^{\circ} q_j \quad (k = n+1, \ldots, n+l)$$

$$Q_k' = -\sum_{j=n+1}^{n+u} \Omega_{kj}^{\circ} q_j^{\cdot} \quad (k = n+l+1, \ldots, n+\mu)$$

$Q_k''$ are holomorphic functions over the set of variables $q_i$, $q_i^{\cdot}$ $(i = 1, \ldots, n)$, $q_j^{\cdot}$ $(j = n+1, \ldots, n+u)$ whose expansions do not contain terms of lower than second order. All $Q_k''$ vanish when $q_j^{\cdot} = 0$ $(j = 1, \ldots, n+u)$, $q_i = 0$ $(i = 1, \ldots, l, s+1, \ldots, n)$ and arbitrary $q_{l+1}, \ldots, q_s$. The zero subscript denotes zero order terms in expansions of related functions.

We shall solve the stability problem of system (2.1) zero solution with respect to all variables $q_j^{\cdot}$ $(j = 1, \ldots, n+u)$ and $q_i$ $(i = 1, \ldots, n)$. We consider systems in which $g_{kj} = g_{kj}^{*}H$ where $H$ is a large positive dimensionless parameter. We denote $H = 1/\mu$ where, as in Sect. 1, $\mu$ is a small parameter, and reduce system (2.1) to the form (1.1). For this we pass to time $\tau = \mu t$ and carry out the transformation

$$z_k = \mu^2 \sum_{j=1}^{n} a_{kj} \frac{dq_j}{d\tau} + \sum_{j=1}^{n} (\mu b_{kj}^{\circ} + g_{kj}^{*\circ}) q_j \quad (k = 1, \ldots, m) \tag{2.2}$$

$$x_k = \sum_{j=1}^{n} a_{kj} \frac{dq_j}{d\tau} \quad (k = 1, \ldots, n), \quad x_k = \sum_{j=n+1}^{n+u} L_{kj} q_j^{\cdot} \; (k = n+1, \ldots, n+u)$$

$$x_{n+u+k} = q_k \quad (k = 1, \ldots, l), \quad x_{n+u-m+k} = q_k \quad (k = s+1, \ldots, n)$$

When $|g_{kj}^{*\circ}|_{k=1,\ldots,m}^{j=l+1,\ldots,s} \neq 0$ the transformation (2.2) is nonsingular and uniformly regular [7]. The determinant $D$ of that transformation is nonzero for any arbitrarily small $\mu$

$$D = |L_{kj}|_{n+1}^{n+u} \, |a_{kj}|_1^n \, |\mu b_{kj}^{\circ} + g_{kj}^{*\circ}|_{k=1,\ldots,m}^{j=l+1,\ldots,s}$$

System (2.1) in new variables assumes the form

$$dz_k / d\tau = Z_k \quad (k = 1, \ldots, m) \tag{2.3}$$

$$\mu^2 \frac{dx_k}{d\tau} = -\sum_{j=1}^{n} (\mu b_{kj}' + g_{kj}') x_j + X_k' + X_k'' \quad (k = 1, \ldots, n)$$

$$\mu \frac{dx_k}{d\tau} = -\sum_{j=1}^{n} \mu B_{kj}' x_j - \sum_{j=n+1}^{n+u} R_{kj}' x_j + X_k' + X_k'' \quad (k = n+1, \ldots, n+u)$$

$$\frac{dx_k}{d\tau} = \sum_{j=1}^{n} d_{kj} x_j \quad (k = n+u+1, \ldots, 2n+u-m)$$

where $X_k'$ $(k = 1, \ldots, n + u)$ are functions $Q_k'$ in new variables, $b_{kj}'$, $g_{kj}'$, $B_{kj}'$ and $R_{kj}'$ are elements of transformed matrices, and $Z_k(z_j, x_i, \mu)$ and $X_k''(z_j, x_i, \mu)$ are holomorphic functions over the set of variables $z_j$, $x_i$ $(j = 1, \ldots, m;\ i = 1, \ldots, 2n + u - m)$ whose expansions do not contain terms of order lower than the second, and which vanish for $x_i = 0$. System (2.3) is of the type of system (1.1). We obtain the degenerate system by setting in Eqs. (2.3) $\mu = 0$. The system

$$\sum_{j=1}^{n} g_{kj}^{\circ} \frac{dq_j}{dt} = Q_k' + \overline{Q}_k'' \quad (k = 1, \ldots, n) \tag{2.4}$$

$$\sum_{j=n+1}^{n+u} R_{kj}^{\circ} q_j^{\cdot} = Q_k' + \overline{Q}_k'' \quad (k = n + 1, \ldots, n + u)$$

where $\overline{Q}_k''$ is the set of nonlinear terms, corresponds to it in old variables. Only those of the nonlinear terms of input equations (2.1) which are caused by gyroscopic forces and forces of generalized mechanical velocities remain in system (2.4).

System (2.4) differs from the systems of precessional equations considered in gyroscopy [8] by the absence of dissipation terms which relate to generalized mechanical velocities.

Applying to system (2.3) the results obtained in Sect. 1, reverting to old variables, and taking into account that the roots of the characteristic equation are invariant with respect to the nonsingular linear transformation, we obtain that when all roots, except the $m$ zero roots, of the system characteristic equation in the first approximation for (2.4), i.e. all roots of equation

$$\left|\begin{array}{c:c:c} \| g_{kj}^{\circ} \| & 0 & \\ \hdashline \| g_{kj}^{\circ}\lambda \| & \| -A_{kj}^{\circ} \| & \\ \hdashline \| g_{kj}^{\circ}\lambda + c_{kj} \| & 0 & \\ \hdashline \| \omega_{kj}^{\circ} \| \;\vdots\; 0 & & \| R_{kj}^{\circ} + \Omega_{kj}^{\circ} \| \\ 0 \;\vdots & & \end{array}\right| \begin{array}{c} 1 \\ \vdots \\ m \\ \vdots \\ s \\ \vdots \\ n \\ \vdots \\ n+u \end{array} = 0 \tag{2.5}$$

have negative real parts, and for $| g_{kj}^{\circ} |_1^n \neq 0$ the equations

$$\left| \| L_{kj}\lambda + R_{kj}^{\circ} + \Omega_{kj}^{\circ} \| \right| \begin{array}{c} n+1 \\ \vdots \\ n+u \end{array} = 0 \tag{2.6}$$

$$\left| \| a_{kj}^{\circ}\beta + g_{kj}^{*\circ} \| \right| \begin{array}{c} 1 \\ \vdots \\ n \end{array} = 0 \tag{2.7}$$

satisfy the Hurwitz conditions, then the zero solution of system (2.4) is stable, and for fairly small $\mu$ the zero solution of system (2.1) is also stable. Let us show that in such case it is sufficient if the equation

$$\left| \| a_{kj}^{\circ}\lambda + b_{kj}^{\circ} + g_{kj}^{\circ} \| \right|_{1}^{n} = 0 \tag{2.8}$$

satisfies Hurwitz's conditions.

In fact, Eq.(2.7) corresponds to Eq.(1.6) in Sect. 1 which is obtained as a degenerate one from the equation (when $\mu = 0$)

$$d_2(\beta, \mu) = \left| \begin{array}{c} \| \delta_{kj}\beta - \mu^2 p_{kj}(\mu) \| \\ \hline \| \delta_{kj}\beta - \mu p_{kj}(\mu) \| \\ \hline \| \delta_{kj}\beta - p_{kj}(\mu) \| \end{array} \right| \begin{array}{c} 1 \\ \vdots \\ \xi \\ \vdots \\ r \\ \vdots \\ n \end{array} = 0$$

If all roots of Eq.(1.6) are imaginary, we consider instead of (1.6) the shortened equation $d_y(\beta, \mu) = 0$, which is obtained from $d_2(\beta, \mu) = 0$, by taking into account in each element of the determinant only terms which contain $\mu$ in powers not greater than the first

$$d_y(\beta, \mu) = \beta^{\xi} \left| \begin{array}{c} \| \delta_{kj}\beta - \mu p_{kj} \| \\ \hline \| \delta_{kj}\beta - p_{kj}(\mu) \| \end{array} \right| \begin{array}{c} \xi+1 \\ \vdots \\ r \\ \vdots \\ n \end{array} = \beta^{\xi} d_3(\beta, \mu) = 0$$

When $\mu \to 0$ the $(n-r)$ roots of equation $d_3(\beta, \mu) = 0$ tend to $\beta_0$ and at the limit are equal to it.

Let us estimate the error in the approximate determination of roots of equation $d_2(\beta, \mu) = 0$ by the roots of the degenerate and shortened equations (roots of equations (1.6) and $d_3(\beta, \mu) = 0$ ). It can be shown that when among the roots of Eq. (1.6) there are no multiples, then for fairly small $\mu$ the corresponding roots of equations $d_2(\beta, \mu) = 0$ and $d_y(\beta, \mu) = 0$ (the $(n-r)$ roots that tend to $\beta_0$ when $\mu \to 0$ ) lie in the root plane on one side of the imaginary axis. Hence, when equation $d_3(\beta, \mu) = 0$ satisfies the Hurwitz conditions, then for fairly small $\mu$ the related $(n-r)$ roots of equation $d_2(\beta, \mu) = 0$ have negative real parts.

Thus, when all roots of Eq. (1.6) are imaginary it is, generally, sufficient to stipulate the fulfilment of Hurwitz conditions by equations (1.4), (1.5), and $d_3(\beta, \mu) = 0$.

In the notation used in system (2.1) Eq. (2.8) corresponds to the equation $d_3(\beta, \mu) = 0$ Hence for the considered systems of gyroscopic stabilization the following statement is valid: let

$$| g_{kj}^{\circ} |_1^n \neq 0, \quad | g_{kj}^{\circ} |_{k=1,\ldots,m}^{j=l+1,\ldots,s} \neq 0$$

If Eq. (2.5) satisfies Hurwitz conditions, i.e. the zero solution of system (2.4) is stable in the first approximation, and Eqs.(2.6) and (2.8) also satisfy those conditions, and the roots of Eqs.(2.6) and (2.7) are simple, then for fairly small parameter $\mu$ (fairly large values of parameter $H$) the zero solution of system (2.1) is stable with respect to all generalized velocities and generalized mechanical coordinates.

System (2.4) admits $m$ holomorphic Liapunov integrals

$$\sum_{j=1}^{n} g_{kj}^\circ q_j + \varphi_k(q_1, \ldots, q_n) = C_k \quad (k = 1, \ldots, m)$$

where $\varphi_k$ is a holomorphic function that does not contain terms of order lower than the second and vanishes for $q_i = 0 \ (i = 1, \ldots, l, s+1, \ldots, n)$ and arbitrary $q_{l+1}, \ldots, q_s$.

## REFERENCES


1. Liapunov, A.M., The general problem of motion stability. Collect. Vol. 2, Moscow, Izd. Akad.Nauk SSSR, 1956.
2. Klimushev, A.I. and Krasovskii, N.N., Uniform asymptotic stability of systems of differential equations with a small parameter in the derivative terms. PMM, Vol. 25, № 4, 1961.
3. Razumikhin, B.S., On the stability of solutions of systems of differential equations with a small coefficient at derivatives. Sib.Matem. Zh., Vol. 4, № 1, 1963.
4. Klimushev, A.I., First approximation stability of nonlinear systems of differential equations with a small parameter in the derivative terms. Tr.Ural'sk.Politekh. Inst., № 211, 1973.
5. Kuz'mina, L.K., On the problem of stability of gyroscopic stabilizers. Tr.Kazansk. Aviats. Inst., Ser.Matem., Mekhan., № 119, 1970.
6. Kuz'mina, L.K., On the stability of gyroscopic stabilization systems. PMM, Vol. 36, № 4, 1972.
7. Lefschetz, S., Geometric Theory of Differential Equations. Moscow, Izd.Inostr.Lit., 1961.
8. Merkin, D.R., Gyroscopic Systems. Moscow, Gostekh-izdat., 1956.


Translated by J.J.D.